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High-order methods for the numerical solution of Volterra integro-differential equations

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Abstract: If a first-order Volterra integro-differential equation is solved by collocation in the space of continuous polynomial splines of degree $m \geqslant 1$, with collocation occurring at the Gauss–Legendre points, then the resulting approximation u converges, at its knots, like $\mathcal{O}(h^{2m})$, while its derivative u' exhibits only $\mathcal{O}(h^m)$-convergence. This paper deals with the question of how to choose the collocation points so that both u and u' converge like $\mathcal{O}(h^{q^*})$, with q^* maximal.

Keywords: Volterra integro-differential equation, collocation by polynomial spline functions, optimal local superconvergence.

1. Introduction

Consider the Volterra integro-differential equation

$$y'(t) = p(t)y(t) + q(t) + \int_0^t K(t, s)y(s)\,\mathrm{d}s, \quad t \in I := [0, T], \tag{1.1}$$

together with the initial condition $y(0) = y_0$. Here, I is a compact interval, and the given functions p, q, and K are assumed to be at least continuous on their respective domains I and $S := \{(t, s): 0 \leqslant s \leqslant t \leqslant T\}$.

In order to discretize the above initial-value problem, let

$$\Pi_N:\ 0 = t_0^{(N)} < t_1^{(N)} < \cdots < t_N^{(N)} = T$$

be a partition of I, and set

$$Z_n := \{t_n^{(N)}: n = 1, \ldots, N-1\}, \qquad \bar{Z}_N := Z_n \cup \{T\},$$
$$\sigma_n^{(N)} := [t_n^{(N)}, t_{n+1}^{(N)}], \quad h_n^{(N)} := t_{n+1}^{(N)} - t_n^{(N)}, \qquad n = 0, \ldots, N-1,$$

and

$$h^{(N)} := \max\{h_n^{(N)}: 0 \leqslant n \leqslant N-1\}, \qquad \bar{h}^{(N)} := \min\{h_n^{(N)}: 0 \leqslant n \leqslant N-1\}.$$

It will be assumed that the sequence of partitions $\{\Pi_N\}$ is quasi-uniform; i.e., there exists a finite constant γ not depending on N such that, for all $N \in \mathbb{N}$,

$$h^{(N)}/\bar{h}^{(N)} \leqslant \gamma. \tag{1.2}$$

For ease of notation we shall subsequently omit the superscript N indicating the dependence on N.

The given problem (1.1) can be related to the partition Π_N by rewriting it in the form

$$y'(t) = p(t)y(t) + q(t) + \int_{t_n}^{t} K(t, s)y(s)\,\mathrm{d}s + F_n(t;\, y), \quad t \in \sigma_n, \tag{1.3}$$

where

$$F_n(t;\, y) := \sum_{i=1}^{n-1} h_i \int_0^1 K(t,\, t_i + \tau h_i)y(t_i + \tau h_i)\,\mathrm{d}\tau \tag{1.4}$$

represents the so-called lag (or: history) term with respect to $t \in \sigma_n$.

The solution of the initial-value problem (1.1) will be approximated in the polynomial spline space

$$S_m^{(0)}(Z_N) := \left\{u \in C(I)\colon\, u|_{\sigma_n} =: u_n \in \pi_m,\ n = 0, \ldots, N-1\right\},$$

where π_m denotes the space of (real) polynomials of degree not exceeding m, with $m \geqslant 1$. Setting, for given parameters $\{c_j\}$ with $0 \leqslant c_1 < \cdots < c_m \leqslant 1$,

$$X_n := \left\{t_{nj} := t_n + c_j h_n\colon\ j = 1, \ldots, m\right\}, \quad n = 0, \ldots, N-1,$$

and

$$X(N) := \bigcup_{n=0}^{N-1} X_n,$$

we seek an element $u \in S_m^{(0)}(Z_n)$ which satisfies (1.1) on the set of collocation points $X(N)$. According to (1.3), the resulting collocation equation assumes the form

$$u_n'(t_{nj}) = p(t_{nj})u_n(t_{nj}) + q(t_{nj}) + h_n \int_0^{c_j} K(t_{nj},\, t_n + \tau h_n)u_n(t_n + \tau h_n)\,\mathrm{d}\tau + F_n(t_{nj};\, u),$$
$$j = 1, \ldots, m, \quad n = 0, \ldots, N-1, \tag{1.5a}$$

with

$$F_n(t;\, u) := \sum_{i=0}^{n-1} h_i \int_0^1 K(t,\, t_i + \tau h_i)u_i(t_i + \tau h_i)\,\mathrm{d}\tau, \quad t \in \sigma_n, \tag{1.5b}$$

approximating the exact lag term (1.4). Moreover, the continuity requirements for the knots Z_N yield the conditions

$$u_n(t_n) = u_{n-1}(t_n), \quad t_n \in Z_N, \tag{1.5c}$$

while the initial condition leads to

$$u_0(t_0) = y(0) = y_0. \tag{1.5d}$$

Let $U_{nj} := u_n'(t_{nj})$, and set

$$L_l(\tau) := \prod_{\substack{k=1\\ k\neq l}}^{m} (\tau - c_k)/(c_l - c_k), \quad l = 1, \ldots, m.$$

We may then write

$$u_n'(t_n + \tau h_n) = \sum_{l=1}^{m} L_l(\tau) U_{nl}, \quad t_n + \tau h_n \in \sigma_n, \tag{1.6a}$$

and

$$u_n(t_n + \tau h_n) = y_n + h_n \sum_{l=1}^{m} \lambda_l(\tau) U_{nl}, \quad t_n + \tau h_n \in \sigma_n; \tag{1.6b}$$

here, we have set $y_n := u_n(t_n)$, and

$$\lambda_l(\tau) := \int_0^{\tau} L_l(v)\,\mathrm{d}v, \quad l = 1, \ldots, m. \tag{1.7}$$

For each $n = 0, \ldots, N-1$, (1.5) represents a linear system in $\mathbb{R}^m$ for the vector $U_n := (U_{n1}, \ldots, U_{nm})^{\mathrm{T}}$; it is easily seen that each system is uniquely solvable whenever the diameter $h := \max_{(n)}(h_n)$ of the partition Π_N is sufficiently small. Once these vectors $\{U_n\}$ have been found, (1.6b) and (1.6a) may be used as interpolation formulas to compute $u(t)$ and $u'(t)$ for any $t \in \sigma_n$.

It is known (see, e.g. [1,2]) that, for $p, q \in C^m(I)$, $K \in C^m(S)$, the global convergence behavior of u on I is described by

$$\| y^{(k)} - u^{(k)} \|_{\infty} = \mathcal{O}(N^{-m}), \quad k = 0, 1,$$

as $N \to \infty$, $Nh \leqslant \gamma T$; this holds for any choice of the collocation parameters $\{c_j\}$ with $0 \leqslant c_1 < \cdots < c_m \leqslant 1$. (Note, incidentally, that $c_1 = 0$ and $c_m = 1$ imply $u \in S_m^{(1)}(Z_n) := S_m^{(0)}(Z_N) \cap C^1(I)$.) Moreover, if the $\{c_j\}$ are given by the zeros of the Legendre polynomial $P_m(2s-1)$ (the Gauss points for (0, 1)), then we obtain local superconvergence on $\overline{Z}_N$,

$$\max_{t_n \in \overline{Z}_N} | y(t_n) - u(t_n) | = \mathcal{O}(N^{-2m}) \quad \text{as } N \to \infty, \quad Nh \leqslant \gamma T,$$

provided the given functions in (1.1) are $2m$ times continuously differentiable.

The purpose of this paper is to provide answers to the following questions:

(i) If we choose for the collocation parameters the Gauss points, what is the largest integer p^* for which

$$\max_{t_n \in \overline{Z}_N} | y'(t_n) - u'(t_n) | = \mathcal{O}(N^{-p^*})$$

holds?

(ii) What is the optimal 'balanced' order of local superconvergence on $\overline{Z}_N$; i.e., what is the largest integer q^* for which we have

$$\max_{t_n \in \overline{Z}_N} | y^{(k)}(t_n) - u^{(k)}(t_n) | = \mathcal{O}(N^{-q^*}), \quad k = 0, 1, \tag{1.8}$$

as $N \to \infty$, $Nh \leqslant \gamma T$?

It will be shown that (i) $p^* = m$, and (ii) $q^* = 2m - 1$; in other words, collocation at the Gauss points does not lead to a local superconvergence result of the form (1.8).

2. Simultaneous local superconvergence for u and u'

Theorem 2.1. *Suppose that the functions p, q, and K in* (1.1) *satisfy $p, q \in C^{2m-1}(I)$, $k \in C^{2m-1}(S)$, with $m \geqslant 1$. If $u \in S_m^{(0)}(Z_N)$ denotes the collocation approximation determined by* (1.5), *and if the collocation parameters* $\{c_j\}$ *are the zeros of the Legendre polynomial* $P_m(2s-1)$ (*i.e., the Gauss points for* $(0, 1)$), *then*

$$\max_{t_n \in \bar{Z}_N} |y(t_n) - u(t_n)| = \mathcal{O}(N^{-2m}), \tag{2.1a}$$

while

$$\max_{t_n \in \bar{Z}_N} |y'(t_n) - u'(t_n)| = \mathcal{O}(N^{-m}), \tag{2.1b}$$

as $N \to \infty$, $Nh \leqslant \gamma T$. *Both estimates are best possible.*

After this negative result (with respect to the problem of generating simultaneously high-order approximations to y and y' on $\bar{Z}_N$), we state the main result on 'balanced' local superconvergence of optimal order; here, the underlying collocation parameters are, not surprisingly, the Radau II points (with $c_m = 1$).

Theorem 2.2. *Let the functions* p, q, *and* K *in* (1.1) *satisfy* $p, q \in C^{2m-2}(I)$, $K \in C^{2m-2}(I)$, *with* $m \geqslant 1$, *and let* $u \in S_m^{(0)}(Z_N)$ *be the collocation approximation given by* (1.5). *If the collocation parameters* $\{c_j\}$ *are the zeros of* $P_m(2s-1) - P_{m-1}(2s-1)$ (*i.e., the Radau II points for* $(0, 1]$), *then we obtain*

$$\max_{t_n \in \bar{Z}_N} |y^{(k)}(t_n) - u^{(k)}(t_n)| = \mathcal{O}(N^{-(2m-1)}), \quad k = 0, 1, \tag{2.2}$$

as $N \to \infty$, $Nh \leqslant \gamma T$. *These estimates are best possible, in the sense that there do not exist points* $\{c_j\}$, *with* $0 \leqslant c_1 < \cdots < c_m \leqslant 1$, *for which the exponent* $2m-1$ *in* (2.2) *can be replaced by* $2m$.

For the sake of completeness we add a result which exhibits the basic principle on which the above theorems are based.

Theorem 2.3. *Let* $0 \leqslant c_1 < \cdots < c_{m-1} < c_m = 1$. *If the functions* p, q, *and* K *in* (1.1) *are sufficiently smooth on their respective domains* I *and* S, *then the collocation approximation* $u \in S_m^{(0)}(Z_N)$ *defined by* (1.5) *satisfies*

$$\max_{t_n \in \bar{Z}_N} |y^{(k)}(t_n) - u^{(k)}(t_n)| = \mathcal{O}(N^{-r}), \quad k = 0, 1, \tag{2.3}$$

as $N \to \infty$, $Nh \leqslant \gamma T$, *with* $m \leqslant r \leqslant 2m-1$. *More precisely, if* d *denotes the degree of precision of the* m*-point interpolatory quadrature formula on* $[0, 1]$ *using the abscissas* $\{c_j\}$, *then* $r = d + 1$.

Since the degree of precision of an m-point interpolatory quadrature formula, subject to the constraint $c_m = 1$, cannot exceed $d^* := 2m - 2$, and since d^* is attained if, and only if, the abscissas $\{c_j\}$ are the Radau II points for $(0, 1]$ (compare, e.g., [5, pp. 103–104]), we obtain Theorem 2.2.

Proofs. One of the crucial ingredients in the proofs of the above results is the representation of the solution of the initial-value problem (1.1) in terms of its resolvent kernel $R(t, s)$. This solution may be expressed in the form

$$y(t) = R(t, 0)y_0 + \int_0^t R(t, s)q(s)\,\mathrm{d}s, \quad t \in I, \tag{2.4}$$

where

$$R(t, s) := 1 + \int_s^t r(t, \tau)\,\mathrm{d}\tau, \quad (t, s) \in S \tag{2.5}$$

(compare [7; 3, pp. 63–65]). Here, the function $r(t, s)$ depends on $p(t)$ and on the kernel $K(t, s)$. To see this, observe that the initial-value problem (1.1) can be rewritten as a Volterra integral equation of the second kind,

$$y(t) = g(t) + \int_0^t k(t, s)y(s)\,\mathrm{d}s, \quad t \in I, \tag{2.6a}$$

where

$$g(t) := y_0 + \int_0^t q(s)\,\mathrm{d}s, \tag{2.6b}$$

and

$$k(t, s) := p(s) + \int_s^t K(\tau, s)\,\mathrm{d}\tau, \quad (t, s) \in S. \tag{2.6c}$$

If $r(t, s)$ denotes the resolvent kernel of the kernel $k(t, s)$ of (2.6a), then it follows from the classical Volterra theory that the (unique) solution of (2.6a) is given by

$$y(t) = g(t) + \int_0^t r(t, s)g(s)\,\mathrm{d}s, \quad t \in I.$$

By Dirichlet's formula this yields, using (2.6b) and (2.6c),

$$\begin{aligned} y(t) &= y_0 + \int_0^t q(s)\,\mathrm{d}s + \int_0^t r(t, s)\left\{y_0 + \int_0^s q(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}s \\ &= \left(1 + \int_0^t r(t, s)\,\mathrm{d}s\right)y_0 + \int_0^t\left(1 + \int_s^t r(t, \tau)\,\mathrm{d}\tau\right)q(s)\,\mathrm{d}s; \end{aligned}$$

by (2.5), this is identical with (2.4).

The error function $e(t) := y(t) - u(t)$ induced by the collocation approximation $u \in S_m^{(0)}(Z_N)$ satisfies, by (1.3) and (1.5a), the integro-differential equation

$$e'(t) = p(t)e(t) + \delta(t) + \int_0^t K(t, s)e(s)\,\mathrm{d}s, \quad t \in I, \tag{2.7}$$

with $e(0) = 0$; here, the so-called defect function δ (compare, e.g. [1]) is defined by

$$\delta(t) := -u'(t) + p(t)u(t) + q(t) + \int_0^t K(t, s)u(s)\,\mathrm{d}s, \quad t \in I. \tag{2.8}$$

Obviously, δ vanishes at all collocation points: $\delta(t_{nj}) = 0$ for $t_{nj} \in X(N)$. Moreover, since we have

$$\| y^{(k)} - u^{(k)} \|_\infty = \mathcal{O}(N^{-m}), \quad k = 0, 1,$$

as $N \to \infty$, $Nh \leqslant \gamma T$ [2], it follows that $\| \delta \|_\infty$ remains uniformly bounded as $N \to \infty$, $Nh \leqslant \gamma T$.

Also, the smoothness hypotheses for p, q, and K imply that δ is smooth on each subinterval σ_n.

By (2.4) the solution of the initial-value problem (2.7) may be written as

$$e(t) = \int_0^t R(t, s)\delta(s)\,\mathrm{d}s, \quad t \in I; \tag{2.9}$$

hence,

$$e'(t) = \delta(t) + \int_0^t R_t(t, s)\mathrm{d}(s)\,\mathrm{d}s, \quad t \in I, \tag{2.10}$$

where we have set $R_t := \partial R/\partial t$.

Consider now (2.9) and (2.10) for $t = t_n \in \overline{Z}_N$: we have

$$e(t_n) = \sum_{i=0}^{n-1} h_i \int_0^1 R(t_n, t_i + \tau h_i)\delta(t_i + \tau h_i)\,\mathrm{d}\tau, \tag{2.11}$$

and

$$e'(t_n) = \delta(t_n) + \sum_{i=0}^{n-1} h_i \int_0^1 R_t(t_n, t_i + \tau h_i)\delta(t_i + \tau h_i)\,\mathrm{d}\tau. \tag{2.12}$$

Assume that the integrals occurring in the above expressions for $e(t_n)$ and $e'(t_n)$ are approximated by m-points interpolatory quadrature formulas employing the abscissas $\{t_{il}\colon l = 1, \ldots, m\}$ (i.e., the quadrature abscissas coincide with the collocation points). If we denote by E_{ni} and $\overline{E}_{ni}$ the corresponding quadrature errors, we obtain (since we have $\delta(t_{il}) = 0$)

$$e(t_n) = \sum_{i=0}^{n-1} h_i E_{ni}, \quad t_n \in \overline{Z}_N, \tag{2.13}$$

and

$$e'(t_n) = \delta(t_n) + \sum_{i=0}^{n-1} h_i \overline{E}_{ni}, \quad t_n \in \overline{Z}_N. \tag{2.14}$$

It follows from the smoothness hypotheses for p, q and K in (1.1), and from the definition of the resolvent kernels $r(t, s)$ and $R(t, s)$ (cf. (2.5)) that the integrands $R(t_n, t_i + \tau h_i) \cdot \delta(t_i + \tau h_i)$ and $R_t(t_n, t_i + \tau h_i) \cdot \delta(t_i + \tau h_i)$ $(0 \leqslant i \leqslant n-1)$ satisfy the same smoothness conditions for $\tau \in [0, 1]$. Moreover, as indicated before, the global convergence (on I) of u to y implies that the defect δ is uniformly bounded as $N \to \infty$, $Nh \leqslant \gamma T$. Hence, the quadrature errors E_{ni} and $\overline{E}_{ni}$ satisfy

$$|E_{ni}| \leqslant C h_i^{d+1}, \qquad |\overline{E}_{ni}| \leqslant \overline{C} h_i^{d+1},$$

where d is the degree of precision of the chosen m-point quadrature formula (this is, of course, an immediate consequence of Peano's kernel theorem). Note, incidentally, that the hypotheses $p, q \in C^{k-1}(I)$, $K \in C^{k-1}(S)$, with $k \geqslant 1$, imply that the solution of (1.1) lies in $C^k(I)$.

Consider now Theorem 2.1: since the above quadrature formulas are the m-point Gauss (–Legendre) formulas, the degree of precision is given by $d = 2m - 1$, and we obtain

$$|E_{ni}| \leqslant C_G h_i^{2m}, \quad |\overline{E}_{ni}| \leqslant \overline{C}_G h_i^{2m}, \qquad 0 \leqslant i \leqslant n-1,$$

with $h_i \leqslant h$, and with $nh \leqslant Nh \leqslant \gamma T$ (this follows from assumption (1.2)). Using these results in (2.13) and (2.14), we obtain the estimates

$$|e(t_n)| \leqslant \gamma T C_G h^{2m}, \quad t_n \in \overline{Z}_N, \tag{2.15a}$$

and

$$|e'(t_n)| \leqslant |\delta(t_n)| + \gamma T \overline{C}_G h^{2m}, \quad t_n \in \overline{Z}_N \tag{2.15b}$$

(as $N \to \infty$, $Nh \leqslant \gamma T$). While (2.15a) is equivalent with (2.1a), it remains to be shown that $|\partial(t_n)| = \mathcal{O}(h^m)$, where m cannot be replaced by an integer greater than m.

To verify this, consider the special case of (1.1) given by

$$y'(t) = \lambda y(t), \quad y(0) = y_0 \neq 0,$$

where λ is a nonvanishing constant. According to (2.8), the corresponding defect function δ has the form

$$\delta_n(t) = -u_n'(t) + \lambda u_n(t), \quad t \in \sigma_n,$$

where the subscript n indicates the restriction of δ to the subinterval σ_n. Since $u \in S_m^{(0)}(Z_N)$, we have $u_n \in \pi_m$, and thus $\delta_n \in \pi_m$. Moreover, $\delta_n(t_{nj}) = 0$, $j = 1, \ldots, m$. These observations imply that $\delta_n(t)$ must be of the form

$$\delta_n(t) = c \cdot \prod_{j=1}^{m} (t - t_{nj}), \quad t \in \sigma_n,$$

with some constant $c \neq 0$. For $t = t_n + h_n$ $(= t_{n+1})$ this yields, using $c_m < 1$,

$$\delta_n(t_{n+1}) = c h_n^m \cdot \prod_{j=1}^{m} (1 - c_j) = \mathcal{O}(h^m),$$

where m cannot be replaced by $m + 1$. Substituting this result in (2.15b) we obtain (2.1b), since $h \leqslant \gamma T N^{-1} = \mathcal{O}(N^{-1})$.

Turning to Theorem 2.3, let now the collocation parameters $\{c_j\}$ satisfy $0 \leqslant c_1 < \cdots < c_{m-1} < c_m = 1$. Since each point $t_n \in \overline{Z}_n$ is then a collocation point, we have $\delta(t_n) = 0$ in (2.15b). This implies, according to the above remarks on the quadrature errors E_{ni} and $\overline{E}_{ni}$, and by the fact that the degree of precision of an m-point interpolatory quadrature formula satisfies $m - 1 \leqslant d \leqslant 2m - 2$, the estimates (2.3), thus proving Theorem 2.3.

The optimal value of d, under the constraint that the last collocation parameter satisfy $c_m = 1$, is given by $d = 2m - 2$: it is attained if, and only if, the $\{c_j\}$ are the Radau II points for the interval (0, 1] (i.e., the zeros of $P_m(2s - 1) - P_{m-1}(2s - 1)$). For this choice we have, in (2.3), $r = d + 1 = 2m - 1$, thus establishing assertion (2.2) of Theorem 2.2. □

The results contained in these three theorems carry over, in a straightforward fashion, to nonlinear Volterra integro-differential equations,

$$y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s))\, ds, \quad t \in I, \qquad y(0) = y_0.$$

Since the derivation of the error equation corresponding to (2.7) will now involve a linearization step (i.e., the application of the Mean-Value Theorem to the functions $f(t, y)$ and $k(t, s, y)$ with respect to their last arguments), the roles of the functions $p(t)$ and $K(t, s)$ will be assumed by $f_y(t, z(t))$ and $k_y(t, s, w(s))$, with z and w denoting suitable values between y and u. Under appropriate smoothness and boundedness assumptions for f_y and k_y the arguments used in the above proofs are readily modified to deal with the nonlinear version of (1.1). We leave the details to the reader.

3. Additional remarks and numerical examples

The practical application of the collocation method (1.5) will, in general, necessitate a further discretization step, namely the approximation of the integrals in (1.5a) and (1.5b) by appropriate numerical integration formulas. If these formulas are chosen so that their degrees of precision are sufficiently high, then the collocation approximation resulting from the fully discretized collocation equation (which we shall denote by $\hat{u}$, in order to emphasize that it will in general differ from the approximation u defined by (1.5) with exact integrals) will exhibit the same order of convergence as u. It has been shown in [2] that if the quadrature abscissas are chosen to be, respectively, $\{t + c_j c_l h_n$: $l = 1,\dots, m\}$ (for the integrals in (1.5a)), and $\{t_{il}$: $l = 1,\dots, m$ (for the integrals in (1.5b)), then this leaves the orders of convergence in (2.1), (2.2), and (2.3) unchanged. Details may be found in [2].

The actual choice of the collocation parameters (Gauss points or Radau II points) will be governed by the following considerations:

(i) If the problem requires generating high-order approximations on $\bar{Z}_N$ only to the solution itself, then one will select the Gauss points; the local superconvergence rate is given by $\mathcal{O}(N^{-2m})$ (Theorem 2.1, (2.1a)).

(ii) If, however, we are interested in high-order approximations on $\bar{Z}_N$ for both y and y', then the Radau II points will be employed, leading to $\mathcal{O}(N^{-(2m-1)})$-convergence (Theorem 2.2, (2.2)).

This choice of the collocation points will be relevant if (1.1) models a physical problem where one is also interested in obtaining good approximations on $\bar{Z}_N$ to the function y' representing an unknown rate of change. Problems of this kind occur above all in population dynamics; see, e.g. [8,4,6] for detailed descriptions of models leading to integro-differential equations of the form (1.1).

We illustrate the results of this paper by two examples.

Example 3.1. See Table 3.1.

$$y'(t) = -\cos(t)y(t) + q(t) - \int_0^t \frac{1+t}{1+s} y(s)\,\mathrm{d}s, \quad t \in [0, 1],$$

with $y(0) = 1$. Here, q is such that the exact solution is given by $y(t) = (1+t)\exp(-t)$.

Table 3.1
Example 3.1; $\hat{u} \in S_2^{(0)}(Z_N)$

N	Gauss points		Radau II points	
	$\lvert\hat{e}(1)\rvert$	$\lvert\hat{e}'(1)\rvert$	$\lvert\hat{e}(1)\rvert$	$\lvert\hat{e}'(1)\rvert$
5	2.75 D−6	1.43 D−3	1.28 D−4	1.12 D−4
10	1.75 D−7	3.31 D−4	1.63 D−5	1.39 D−5
20	1.10 D−8	7.96 D−5	2.05 D−6	1.73 D−6
40	6.90 D−10	1.95 D−5	2.57 D−7	2.15 D−7

Table 3.2
Example 3.2; $\hat{u} \in S_2^{(0)}(Z_N)$

N	Gauss points		Radau II points	
	$\lvert\hat{e}(1)\rvert$	$\lvert\hat{e}'(1)\rvert$	$\lvert\hat{e}(1)\rvert$	$\lvert\hat{e}'(1)\rvert$
5	1.73 D−4	1.47 D−1	2.58 D−3	7.08 D−3
10	1.12 D−5	4.07 D−2	3.30 D−4	9.00 D−4
20	7.06 D−7	1.07 D−2	4.14 D−5	1.13 D−4
40	4.43 D−8	2.76 D−3	5.17 D−6	1.41 D−5

Example 3.2 (Pouzet [9]). See Table 3.2.

$$y'(t) = -y(t) + 1 + 2t + \int_0^t t(1+2t)\exp(s(t-s))y(s)\,\mathrm{d}s, \quad t \in [0, 1],$$

with $y(0) = 1$. The exact solution is $y(t) = \exp(t^2)$.

These integro-differential equations were solved by collocation in the polynomial spline space $S_2^{(0)}(Z_N)$, with the underlying meshes being uniform. The integrals in the collocation equation (1.5a), (1.5b) were approximated by two-point interpolatory quadrature formulas of the types described in the first paragraph of this section (see also [2]). Accordingly, the convergence behavior of the resulting approximation $\hat{u}$ on $\bar{Z}_N$ is described by

$$\max_{t_n \in \bar{Z}_N} |y(t_n) - \hat{u}(t_n)| = \mathcal{O}(N^{-4}), \qquad \max_{t_n \in \bar{Z}_N} |y'(t_n) - \hat{u}'(t_n)| = \mathcal{O}(N^{-2}),$$

if $c_1 = \frac{1}{6}(3 - \sqrt{3})$, $c_2 = \frac{1}{6}(3 + \sqrt{3})$ (Gauss points); and by

$$\max_{t_n \in \bar{Z}_N} |y^{(k)}(t_n) - \hat{u}^{(k)}(t_n)| = \mathcal{O}(N^{-3}), \quad k = 0, 1,$$

if $c_1 = \frac{1}{3}$, $c_2 = 1$ (Radau II points).

A sample of numerical results is displayed in Tables 3.1 and 3.2; here, $\hat{e}(t) := y(t) - \hat{u}(t)$.

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